

Algebraic Proof of the Symmetric Space Theorem

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Abstract

I give a relatively elementary proof of the symmetric space theorem, due to Goddard, Nahm and Olive [1]. Unlike their original proof, which involves the quark-model construction, I only use elementary algebraic techniques.

I. Introduction

In 1985 Goddard, Nahm and Olive [1] proved an interesting theorem, which has been called the *symmetric space theorem*. It relates the vanishing of certain coset Virasoro algebras to the existence of symmetric spaces. This has further interesting mathematical consequences in representation theory of affine Kac-Moody algebras, since the vanishing of the coset Virasoro algebra is the condition for finite reducibility of representations of affine algebras restricted to certain affine subalgebras.

The original proof by the above authors involved the use of physical concepts, such as quarks. The purpose of the present paper is to state and prove the symmetric space theorem, by using purely algebraic concepts. In particular, I shall carry out the proof following an idea used by Witt [2], thereby using only elementary algebraic transformations.

In section II. I introduce relevant definitions and notations. In section III. I first prove three auxiliary lemmas and two corollaries, then state and prove the theorem.

II. Definitions and Notations

II.1. Affine Algebras

The *affine Kac-Moody algebra*, affine algebra for short,

$$\tilde{\mathcal{L}}(\mathfrak{g}) = (\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \mathbb{C}\mathcal{K} \quad (1)$$

associated to an underlying simple Lie algebra \mathfrak{g} is defined by the following commutation relations:

$$[t^m \otimes x, t^n \otimes y] = t^{m+n} \otimes [x, y] + m \delta_{m,-n} (x|y) \mathcal{K} \quad (2)$$

where \mathcal{K} spans the center of $\tilde{\mathcal{L}}(\mathfrak{g})$ and is called the *central term*, and $(\cdot|\cdot)$ is the *normalized invariant form* on \mathfrak{g} . The term invariant means that

$$([x, y]|z) = (x|[y, z]) \quad \forall x, y, z \in \mathfrak{g}. \quad (3)$$

It is well known (see for example [3]), that two invariant symmetric bilinear forms on a simple Lie algebra differ only by a scalar factor. The normalized invariant form, which will be used henceforth, is defined such that the highest root has length $\sqrt{2}$.

The affine algebra for an underlying abelian algebra \mathfrak{g} can be defined similarly, by (1) and (2). However, for abelian \mathfrak{g} any nondegenerate symmetric form $(\cdot|\cdot)$ is invariant. In contrast to the case of a simple algebra, such a form cannot be further determined by algebraic constraints derived from the Lie algebra structure. In this article abelian algebras will almost always appear as subalgebras of the simple algebra $so(n)$. Therefore, there will be a natural choice of $(\cdot|\cdot)$, namely the restriction of the normalized invariant form of $so(n)$ to \mathfrak{g} . In the definitions, where an abelian algebra is not given as a subalgebra of $so(n)$, $(\cdot|\cdot)$ may be any fixed nondegenerate symmetric form.

When the central element \mathcal{K} of the affine algebra $\tilde{\mathcal{L}}(\mathfrak{g})$ acts as a scalar k id in a representation, then k is called the *level of this representation*.

For a reductive Lie algebra

$$\mathfrak{u} = \mathfrak{u}_0 \oplus \mathfrak{u}_1 \oplus \dots \oplus \mathfrak{u}_S \quad (4)$$

with center \mathfrak{u}_0 and simple ideals $\mathfrak{u}_1, \dots, \mathfrak{u}_S$, the affine algebra $\tilde{\mathcal{L}}(\mathfrak{u})$ is the direct sum of the affine algebras associated to the ideals $\mathfrak{u}_0, \dots, \mathfrak{u}_S$:

$$\tilde{\mathcal{L}}(\mathfrak{u}) = \bigoplus_{s=0}^S \tilde{\mathcal{L}}(\mathfrak{u}_s) = \bigoplus_{s=0}^S ((\mathbb{C}[t, t^{-1}] \otimes \mathfrak{u}_s) \oplus \mathbb{C}\mathcal{K}_s). \quad (5)$$

The affine algebra associated to a finite-dimensional reductive Lie algebra \mathfrak{u} is thus a S -dimensional central extension of the *loop algebra* $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{u}$ associated to \mathfrak{u} . (In the case of an underlying semi-simple Lie algebra the affine algebra is indeed the universal central extension of the loop algebra.) The loop algebra can be identified with the Lie algebra of polynomial maps of S^1 into \mathfrak{u} .

II.2. The Virasoro Algebra and the Sugawara Construction

The *Virasoro algebra*

$$\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d_n \oplus \mathbb{C} c$$

is defined by the following commutation relations:

$$\begin{aligned} [c, d_n] &= 0 \\ [d_m, d_n] &= (m - n)d_{m+n} + \delta_{m, -n} \frac{m^3 - m}{12} c \quad \forall m, n \in \mathbb{Z}. \end{aligned} \quad (6)$$

Let \mathfrak{g} be a simple Lie algebra. Let $\{u_i\}$ and $\{u^i\}$ be dual bases of \mathfrak{g} . The *quadratic Casimir operator* is the following element of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of \mathfrak{g} :

$$\Omega = \sum_{i=1}^{\dim \mathfrak{g}} u_i u^i. \quad (7)$$

Ω commutes with any element of \mathfrak{g} , therefore, by Schur lemma it acts as a scalar, denoted by ω_ρ , on any irreducible representation ρ of \mathfrak{g} . Given a representation of the affine algebra $\tilde{\mathcal{L}}(\mathfrak{g})$ of level $k \neq -\frac{1}{2}\omega_{ad}$, where ω_{ad} denotes the scalar value of the quadratic Casimir operator in the adjoint representation, one can construct a representation of Vir by using the following *Sugawara operators* (cf. [4]), where the notation $x^{(n)} \equiv t^n \otimes x$ is used:

$$\begin{aligned} L_0 &:= \frac{1}{(2k + \omega_{ad})} \left(\sum_i u_i u^i + 2 \sum_{n=1}^{\infty} \sum_i u_i^{(-n)} u^{i(n)} \right) \\ L_n &:= \frac{1}{(2k + \omega_{ad})} \sum_{m \in \mathbb{Z}} \sum_i u_i^{(-m)} u^{i(m+n)} \quad \forall n \in \mathbb{Z} \setminus \{0\}. \end{aligned} \quad (8)$$

The operators L_n obey the commutation relations (6), where the central element of the Virasoro algebra takes the scalar value

$$c = \frac{2k \dim \mathfrak{g}}{2k + \omega_{ad}}. \quad (9)$$

For an underlying abelian algebra the same equations hold with $\omega_{ad} = 0$. In this case (9) yields $c = \dim \mathfrak{g}$. The Sugawara operators L_m and the elements $x^{(n)}$ of the affine algebra obey the following commutation relations:

$$[L_m, x^{(n)}] = -nx^{(m+n)} \quad \forall m, n \in \mathbb{Z}, x \in \mathfrak{g}. \quad (10)$$

For a reductive Lie algebra as in (4), the Sugawara operators corresponding to $\tilde{\mathcal{L}}(\mathfrak{u})$ in a representation of levels $k_s \neq -\frac{1}{2}\omega_{ad_s}$ are given by

$$L_n^{\mathfrak{u}} := \sum_{s=0}^S L_n^{\mathfrak{u}_s},$$

where $L_n^{\mathfrak{u}_s}$ is the Sugawara operator for the individual ideal \mathfrak{u}_s . The $L_n^{\mathfrak{u}_s}$ provide a representation of the Virasoro algebra with central value

$$c_{\mathfrak{u}} = \sum_{s=0}^S \frac{2k_s \dim \mathfrak{u}_s}{2k_s + \omega_{ad_s}}. \quad (11)$$

Again, the $L_n^{\mathfrak{u}}$ satisfy a commutation relation similar to (10), where $x \in \mathfrak{u}$.

II.3. The Coset Construction

Let U be a compact Lie subgroup of the orthogonal group $SO(n)$. Then the complexified Lie algebra of U is a reductive algebra \mathfrak{u} as in (4), which is a subalgebra of the orthogonal algebra $so(n)$.

The inclusion of \mathfrak{u} in $so(n)$ gives rise to a homomorphism of the associated affine algebras:

$$\tilde{\mathcal{L}}(\mathfrak{u}) = \bigoplus_{s=0}^S ((\mathbb{C}[t, t^{-1}] \otimes \mathfrak{u}_s) \oplus \mathbb{C}\mathcal{K}_s) \rightarrow \tilde{\mathcal{L}}(so(n)) = (\mathbb{C}[t, t^{-1}] \otimes so(n)) \oplus \mathbb{C}\mathcal{K}. \quad (12)$$

The obvious inclusion homomorphism of the loop algebras is lifted consistently to a homomorphism of the affine algebras in (12) by letting

$$\mathcal{K}_s \mapsto j_s \mathcal{K} \quad \text{for } s = 0, \dots, S, \quad (13)$$

where for the simple ideals \mathfrak{u}_s , $s \geq 1$, the factor j_s is the *Dynkin index*, which is defined as the ratio of the normalized invariant form $(\cdot|\cdot)_{so(n)}$ on $so(n)$ (restricted to \mathfrak{u}_s) and the normalized invariant form $(\cdot|\cdot)_s$ on \mathfrak{u}_s i.e.

$$(x|y)_{so(n)} = j_s (x|y)_s \quad \forall x, y \in \mathfrak{u}_s. \quad (14)$$

For $s = 0$ we choose $(\cdot|\cdot)_0$ to be the restriction of $(\cdot|\cdot)_{so(n)}$ to \mathfrak{u}_0 . Then, by letting $j_0 = 1$, equation (14) holds also for $s = 0$.

In this situation the Sugawara construction can be applied to both $so(n)$ and \mathfrak{u} to obtain Sugawara operators $L_n^{so(n)}$ and $L_n^{\mathfrak{u}}$ which are different in general and form representations of Vir. We can calculate their central values by using (9) and (11). We get

$$c_{so(n)} = \frac{k_{so(n)} n (n-1)}{2k_{so(n)} + 2n - 4} \quad (15)$$

and

$$c_{\mathfrak{u}} = \sum_{s=0}^S \frac{2k_{so(n)} j_s \dim \mathfrak{u}_s}{2k_{so(n)} j_s + \omega_{ad_s}}, \quad (16)$$

respectively. In (15) I substituted $\dim(so(n)) = n(n-1)/2$ and

$$\omega_{ad_{so(n)}} = 2n - 4. \quad (17)$$

(The expression (17) can be computed, for example, by using the relation $\omega_{ad} = 2\check{h}$, where \check{h} is the *dual Coxeter number*. A table of dual Coxeter numbers can be found in [4].) In (16) I used $k_s = j_s k_{so(n)}$, which follows from (13).

By (10) we have

$$[L_l^{so(n)}, x^{(m)}] = -mx^{(m+l)} \quad \forall l, m \in \mathbb{Z}, x \in so(n). \quad (18)$$

$$[L_l^{\mathfrak{u}}, x^{(m)}] = -mx^{(m+l)} \quad \forall l, m \in \mathbb{Z}, x \in \mathfrak{u}. \quad (19)$$

Therefore, the difference operators

$$K_l := L_l^{so(n)} - L_l^{\mathfrak{u}}$$

commute with each $x^{(m)} \in \tilde{\mathcal{L}}(\mathfrak{u})$ individually

$$[K_l, x^{(m)}] = 0 \quad \forall l, m \in \mathbb{Z}, x \in \mathfrak{u}, \quad (20)$$

and consequently, by (8), with the corresponding Sugawara operator

$$[K_l, L_m^u] = 0 \quad \forall l, m \in \mathbb{Z}. \quad (21)$$

It follows, that

$$[K_l, K_m] = [L_l^{so(n)}, L_m^{so(n)}] - [L_l^u, L_m^u] \quad \forall l, m \in \mathbb{Z}. \quad (22)$$

We deduce that the K_m , like the $L_m^{so(n)}$ and the L_m^u , define a representation of Vir, whose central charge is equal to the difference of the Sugawara values

$$\begin{aligned} c_K &= c_{so(n)} - c_u \\ &= \frac{k_{so(n)} n (n-1)}{2k_{so(n)} + 2n - 4} - \sum_{s=0}^S \frac{2k_{so(n)} j_s \dim \mathfrak{u}_s}{2k_{so(n)} j_s + \omega_{ad_s}}. \end{aligned} \quad (23)$$

The above construction of representations of the Virasoro algebra is called the *coset construction*. It was introduced by Goddard and Olive in [5]. It can be applied similarly for any simple (or even reductive) Lie algebra \mathfrak{g} instead of $so(n)$ and a reductive subalgebra $\mathfrak{u} \subseteq \mathfrak{g}$, but in this article I shall only consider the case $\mathfrak{g} = so(n)$.

An important property of all the above constructions is, that they preserve the unitarity of the involved representations i.e. a unitary representation of the affine algebra $\tilde{\mathcal{L}}(so(n))$, induces a unitary representation of its subalgebra $\tilde{\mathcal{L}}(\mathfrak{u})$. Then by the Sugawara construction applied to the unitary representations of $\tilde{\mathcal{L}}(\mathfrak{u})$ and $\tilde{\mathcal{L}}(so(n))$ we get two different unitary representations of Vir. Finally, the resulting coset representation of Vir is again unitary.

The Virasoro algebra has no nontrivial unitary representations with zero central charge, so that the coset Virasoro algebra vanishes iff $c_K = 0$. Furthermore, as was first shown in [5], the representation of $\tilde{\mathcal{L}}(\mathfrak{u})$ induced by a unitary highest weight representation of $\tilde{\mathcal{L}}(so(n))$ is finitely reducible, if and only if the coset Virasoro algebra vanishes.

It can be shown that $c_K = 0$ is only possible for level 1 representations of $\tilde{\mathcal{L}}(so(n))$, (see e.g. [6]). In this case (23) reduces to

$$c_K = \frac{n}{2} - \sum_{s=0}^S \frac{2j_s \dim \mathfrak{u}_s}{2j_s + \omega_{ad_s}} \quad \text{for } k_{so(n)} = 1. \quad (24)$$

II.4. Indices of Representations

The *index* κ_ρ of a representation ρ of a simple Lie algebra is defined as the ratio between the trace form of the representation and the normalized invariant form on the algebra, i.e.

$$\text{Tr}(\rho(x)\rho(y)) = \kappa_\rho (x|y). \quad (25)$$

For a simple Lie algebra \mathfrak{g} or an abelian subalgebra of a simple Lie algebra in an n -dimensional representation ρ of \mathfrak{g} , on which the Casimir operator is a scalar multiple of the identity, $\rho(\Omega) = \omega_\rho \text{id}_n$ one gets by using (7) and (25):

$$\kappa_\rho \dim \mathfrak{g} = \omega_\rho n. \quad (26)$$

In particular, if ρ is the adjoint representation of a simple Lie algebra, then (26) yields for its index

$$\kappa_{ad} = \omega_{ad}. \quad (27)$$

This equation also holds (trivially) for an abelian Lie algebra, since in this case $\kappa_{ad} = \omega_{ad} = 0$.

It can be shown, that the index of the natural representation of $so(n)$ (i.e. the representation of $so(n)$ by antisymmetric $n \times n$ -matrices) is 2 (see e.g. [6]). Let \mathfrak{u} be a reductive subalgebra of $so(n)$ as in subsection II.3.. Then the index κ_{ρ_s} of the representation ρ_s of the ideal \mathfrak{u}_s obtained by restricting the natural representation of $so(n)$ to \mathfrak{u}_s is determined by

$$\text{Tr}(\rho_s(x)\rho_s(y)) = 2 (x|y)_{so(n)} = 2 j_s (x|y)_s \quad \forall x, y \in \mathfrak{u}_s$$

i.e.

$$\kappa_{\rho_s} = 2 j_s. \quad (28)$$

Note, that with our choice of $(\cdot|\cdot)_0$ the definition of the index makes sense also for the abelian subalgebra \mathfrak{u}_0 in the representation ρ_0 and we get $\kappa_0 = 2$.

II.5. Infinitesimal Symmetric Spaces

Let \mathfrak{g} be a semi-simple Lie algebra and let σ be an *involution* of \mathfrak{g} , i.e. an automorphism of \mathfrak{g} of order 2: $\sigma^2 = \text{id}$. Let

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

be the decomposition of \mathfrak{g} into eigenspaces of σ

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid \sigma(x) = x\}, \quad \mathfrak{g}_1 = \{x \in \mathfrak{g} \mid \sigma(x) = -x\},$$

then $\mathfrak{g}/\mathfrak{g}_0 \cong \mathfrak{g}_1$ is called an *infinitesimal symmetric space*.

Under the above conditions the following relations hold

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1, \quad [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_0. \quad (29)$$

This means that the Lie algebra \mathfrak{g} is $\mathbb{Z}/2\mathbb{Z}$ -graded. On the other hand, given a $\mathbb{Z}/2\mathbb{Z}$ -gradation on \mathfrak{g} (i.e. a decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that (29) holds), then an involution σ of \mathfrak{g} is defined by letting $\sigma(x) = x$ for all $x \in \mathfrak{g}_0$ and $\sigma(x) = -x$ for all $x \in \mathfrak{g}_1$. Thus the infinitesimal symmetric spaces can equivalently be defined by (29).

Since the Killing form is invariant under automorphisms, we have

$$\begin{aligned} Tr(ad(x)ad(y)) &= Tr(ad(\sigma(x))ad(\sigma(y))) \\ &= -Tr(ad(x)ad(y)) \quad \forall x \in \mathfrak{g}_0, y \in \mathfrak{g}_1. \end{aligned} \quad (30)$$

Thus \mathfrak{g}_0 and \mathfrak{g}_1 are orthogonal with respect to the Killing form on \mathfrak{g} and the restriction of the Killing form to \mathfrak{g}_0 respectively to \mathfrak{g}_1 is nondegenerate.

The list of all symmetric spaces is due to E. Cartan. A derivation based on Kac' classification of finite order automorphisms of semisimple Lie algebras can be found in [7], cf. also [1].

II.6. The underlying real representation of an orthogonal representation

A complex representation of a compact Lie group U is called *orthogonal* if there exists a nondegenerate symmetric bilinear form $(\cdot|\cdot)$ on the representation space \mathfrak{p} , such that

$$(g(x)|g(y)) = (x|y) \quad \forall g \in U, x, y \in \mathfrak{p}, \quad (31)$$

i.e. the form is *invariant* under U . By choosing an orthonormal base of \mathfrak{p} with respect to $(\cdot|\cdot)$ we get a matrix representation of U by orthogonal matrices. If U is connected and the orthogonal representation is faithful this gives an embedding of U in $SO(n)$.

On the Lie algebra level the invariance property is the following

$$(u(x)|y) = -(x|u(y)) \quad \forall u \in \mathfrak{u}, x, y \in \mathfrak{p}, \quad (32)$$

for example (3) is equivalent to this equation in the special case of the adjoint representation. Since the Killing form is invariant under the adjoint action and since it is nondegenerate for a semisimple Lie algebra, the adjoint representation of a semisimple Lie algebra \mathfrak{g} is an orthogonal representation.

In the case of a $\mathbb{Z}/2\mathbb{Z}$ -graded Lie algebra, the representation of \mathfrak{g}_0 on \mathfrak{g}_1 is an orthogonal representation, the restriction of the Killing form to \mathfrak{g}_1 being an invariant form.

It is well known (see e.g. [8, Prop.(6.4)]), that every complex orthogonal representation of a compact Lie group U is the complexification of a real representation of U . This means, that there exists a real subspace $\mathfrak{p}^{\mathbb{R}}$ of \mathfrak{p} , invariant under the group action ($U(\mathfrak{p}^{\mathbb{R}}) \subseteq \mathfrak{p}^{\mathbb{R}}$), such that

$$\mathfrak{p} = \mathfrak{p}^{\mathbb{R}} \oplus i \mathfrak{p}^{\mathbb{R}}$$

as a real vector space, (i.e. each element of $p \in \mathfrak{p}$ can be uniquely decomposed as $p = x + iy$ with $x, y \in \mathfrak{p}^{\mathbb{R}}$) and such that

$$u(x + iy) = u(x) + i u(y) \quad \forall u \in U, x, y \in \mathfrak{p}^{\mathbb{R}}.$$

Although $\mathfrak{p}^{\mathbb{R}}$ is in general not uniquely determined as a real subspace of \mathfrak{p} , its isomorphic type as a real U -module is uniquely determined by the isomorphic type of the complex U -module \mathfrak{p} , since \mathfrak{p} is isomorphic to $\mathfrak{p}^{\mathbb{R}} \oplus \mathfrak{p}^{\mathbb{R}}$ as a real U -module.

III. The Symmetric Space Theorem

The following assumptions and notations shall be valid for the whole section: *Let U be a compact Lie group and let the reductive Lie algebra $\mathfrak{u} = \mathfrak{u}_0 \oplus \mathfrak{u}_1 \oplus \dots \oplus \mathfrak{u}_S$ with center \mathfrak{u}_0 and simple ideals $\mathfrak{u}_1, \dots, \mathfrak{u}_S$ be the complexification of the Lie algebra of U . Let $\{u^i\}_{i=1, \dots, \dim \mathfrak{u}}$ be a base for \mathfrak{u} consisting of antihermitian elements, which is the union of bases $\{u^i\}_{i \in I_s}$ for the ideals \mathfrak{u}_s such that $(u^i|u^j) = -\delta_{ij}$ for $i, j \in I_s$. (This is possible since on the real subalgebra of antihermitian elements the normalized invariant form is negative definite.) For $i \in \{1, \dots, \dim \mathfrak{u}\}$ let $s(i)$ denote the index of the ideal, which contains the element u^i , $u^i \in \mathfrak{u}_{s(i)}$, such that $0 \leq s(i) \leq S$.*

Before stating the theorem I shall prove three lemmas, which will be needed for its proof, but can also be useful by themselves.

Lemma 1 *The following equation holds in the universal enveloping algebra $\mathfrak{U}(\mathfrak{u})$ and thus in any representation of \mathfrak{u}*

$$2 \sum_{i \in I_s} u^i u^j u^i = -2\Omega_s u^j + \delta_{s(j),s} \omega_{ad_s} u^j \quad \text{for } 0 \leq s \leq S, \quad 1 \leq j \leq \dim \mathfrak{u}, \quad (33)$$

where Ω_s denotes the quadratic Casimir operator of the ideal \mathfrak{u}_s .

Proof: The quadratic Casimir operator of \mathfrak{u}_s can be written as

$$\Omega_s = - \sum_{i \in I_s} (u^i)^2.$$

Applying $ad_{\mathfrak{u}}(\Omega_s)$ to u^j gives $\omega_{ad_s} u^j$, if $u^j \in \mathfrak{u}_s$ and 0 otherwise, since $[u^i, u^j] = 0$ for $s(i) \neq s(j)$. Therefore,

$$\begin{aligned} ad_{\mathfrak{u}}(\Omega_s) u^j &= \delta_{s(j),s} \omega_{ad_s} u^j = - \sum_{i \in I_s} ad(u^i)^2(u^j) = - \sum_{i \in I_s} [u^i [u^i, u^j]] \\ &= - \sum_{i \in I_s} (u^i (u^i u^j - u^j u^i) - (u^i u^j - u^j u^i) u^i) \\ &= - \sum_{i \in I_s} ((u^i)^2 u^j - u^i u^j u^i - u^i u^j u^i + u^j (u^i)^2) \\ &= 2\Omega_s u^j + 2 \sum_{i \in I_s} u^i u^j u^i, \end{aligned}$$

where we used, that Ω_s commutes with u^j . □

Applying (33) again to u^j and summing over j we immediately get

Corollary 1 *The sum of the elements $(u^i u^j)^2$ lies in the center of $\mathfrak{U}(\mathfrak{u})$ and is given in terms of the quadratic Casimir operators as follows:*

$$2 \sum_{i \in I_s} \sum_{j \in I_t} (u^i u^j)^2 = (2\Omega_t - \delta_{st} \omega_{ad_s}) \Omega_s \quad \text{for } 0 \leq s, t \leq S. \quad (34)$$

By taking the trace of equation (34) we immediately get the following corollary, which will be used in the proof of the next lemma :

Corrolary 2 *In any n -dimensional matrix representation of \mathfrak{u} , such that the operators u^i are represented by matrices M^i and such that the quadratic Casimir operator Ω_s of the ideal \mathfrak{u}_s acts as a scalar $\omega_s \text{id}_n$ for $s = 0, \dots, S$, the following identity holds:*

$$2 \sum_{i \in I_s} \sum_{j \in I_t} \text{Tr}((M^i M^j)^2) = (2\omega_t - \delta_{st} \omega_{ad_s}) \omega_s n \quad \text{for } 0 \leq s, t \leq S. \quad (35)$$

The following further assumptions and notations shall again be valid in the sequel:

Let \mathfrak{p} be a faithful n -dimensional orthogonal complex representation of the compact group U . This defines an embedding of U into $SO(n)$ and we shall thus view U as a subgroup of $SO(n)$. Let $\{p_\alpha\}_{\alpha=1, \dots, n}$ be an orthonormal base of $\mathfrak{p}^{\mathbb{R}}$. Let the application of the operators u^i on the basis p_α be described by the real antisymmetric matrices M^i , as follows

$$u^i(p_\alpha) = \sum_{\gamma=1}^n M_{\gamma\alpha}^i p_\gamma, \quad (36)$$

Furthermore let y_s denote the index of the representation of \mathfrak{u}_s on $\mathfrak{u} \oplus \mathfrak{p}$.

To evaluate y_s we note that (a) the index of a direct sum of representations is the sum of the indices of the direct summands and (b) the index of the adjoint action of \mathfrak{u}_s on \mathfrak{u} is ω_{ad_s} , because of (27) and since the adjoint action of \mathfrak{u}_s on the sum of the \mathfrak{u}_t with $t \neq s$ is trivial. Denoting the index of the representation of the ideal \mathfrak{u}_s on \mathfrak{p} by κ_s (instead of κ_{ρ_s}), we get

$$y_s = \omega_{ad_s} + \kappa_s = \omega_{ad_s} + 2j_s. \quad (37)$$

It follows, that $y_s > 0$ for $s = 0, \dots, S$.

We assume, that the underlying real representation $\mathfrak{p}^{\mathbb{R}}$ of \mathfrak{p} is irreducible. This holds for example if \mathfrak{p} is itself irreducible, but \mathfrak{p} can also decompose as $\mathfrak{p} \cong \mathfrak{q} \oplus \mathfrak{q}^*$, where \mathfrak{q} is some irreducible complex representation, such that $\mathfrak{q} \not\cong \mathfrak{q}^*$. (The theorem also holds when $\mathfrak{p}^{\mathbb{R}}$ is reducible, and it can be derived from the irreducible case (see e.g. [6]).)

It can be shown that under the assumption, that $\mathfrak{p}^{\mathbb{R}}$ is irreducible as a U -module, the Casimir operators Ω_s act as real positive scalars on \mathfrak{p} , although \mathfrak{p} is not irreducible as a module over \mathfrak{u}_s ,

$$\rho(\Omega_s) = \omega_s \text{id}_n \quad \text{with } \omega_s > 0. \quad (38)$$

Therefore, (26) and (35) hold in this situation.

Lemma 2 *Let V be a level one representation space of the affine algebra $\tilde{\mathcal{L}}(so(n))$. Let c_u denote the central element of the Virasoro algebra constructed from $\tilde{\mathcal{L}}(\mathfrak{u})$ by the Sugawara construction on V and let c_K denote the central element of the coset Virasoro algebra on V . Furthermore, let*

$$J_{\alpha\beta\gamma\delta} := \sum_{s=0}^S \frac{1}{y_s} \sum_{i \in I_s} (M_{\gamma\beta}^i M_{\delta\alpha}^i + M_{\beta\alpha}^i M_{\delta\gamma}^i + M_{\alpha\gamma}^i M_{\delta\beta}^i) , \quad (39)$$

where M is defined in (36), then the following identity holds

$$\sum_{\alpha\beta\gamma\delta} (J_{\alpha\beta\gamma\delta})^2 = \frac{6}{n} c_u c_K. \quad (40)$$

Proof: First let us bring c_K and c_u in the form, that is most adequate to the following calculations. Substituting (28) in (24) we get

$$c_K = \frac{n}{2} - \sum_{s=0}^S \frac{\kappa_s \dim \mathfrak{u}_s}{\kappa_s + \omega_{ads}} \stackrel{(26)(37)}{=} \frac{n}{2} - \sum_{s=0}^S \frac{n\omega_s}{y_s}. \quad (41)$$

For c_u alone we have

$$c_u = \sum_{s=0}^S \frac{n\omega_s}{y_s} > 0. \quad (42)$$

The sums

$$J_{\alpha\beta\gamma\delta}^i := M_{\gamma\beta}^i M_{\delta\alpha}^i + M_{\beta\alpha}^i M_{\delta\gamma}^i + M_{\alpha\gamma}^i M_{\delta\beta}^i , \quad (43)$$

are invariant under cyclic permutation of the indices α, β, γ : $J_{\alpha\beta\gamma\delta}^i = J_{\gamma\alpha\beta\delta}^i = J_{\beta\gamma\alpha\delta}^i$, so that

$$\begin{aligned} \frac{1}{3} \sum_{\alpha,\beta,\gamma,\delta} J_{\alpha\beta\gamma\delta}^i J_{\alpha\beta\gamma\delta}^j &= \sum_{\alpha,\beta,\gamma,\delta} J_{\alpha\beta\gamma\delta}^i M_{\gamma\beta}^j M_{\delta\alpha}^j \\ &= \sum_{\alpha,\beta,\gamma,\delta} (M_{\gamma\beta}^i M_{\delta\alpha}^i + M_{\beta\alpha}^i M_{\delta\gamma}^i + M_{\alpha\gamma}^i M_{\delta\beta}^i) M_{\gamma\beta}^j M_{\delta\alpha}^j \\ &= (Tr(M^i M^j))^2 - 2Tr((M^i M^j)^2) \\ &= \delta_{ij} \kappa_{s(i)}^2 - 2Tr((M^i M^j)^2) , \end{aligned} \quad (44)$$

where κ_s is the index of the representation of \mathfrak{u}_s on \mathfrak{p} . For the third equality I used the antisymmetry of the matrices M^i . Using (44) we get

$$\begin{aligned}
\frac{1}{3} \sum_{\alpha, \beta, \gamma, \delta} (J_{\alpha\beta\gamma\delta})^2 &= \frac{1}{3} \sum_{s,t=0}^S \frac{1}{y_s y_t} \sum_{i \in I_s, j \in I_t} \sum_{\alpha, \beta, \gamma, \delta} J_{\alpha\beta\gamma\delta}^i J_{\alpha\beta\gamma\delta}^j \\
&\stackrel{(44)}{=} \sum_{s=0}^S \frac{\kappa_s^2 \dim \mathfrak{u}_s}{y_s^2} - \sum_{s,t=0}^S \frac{2}{y_s y_t} \sum_{i \in I_s, j \in I_t} \text{Tr}((M^i M^j)^2) \\
&\stackrel{(26)(35)}{=} \sum_{s=0}^S \left(\frac{\kappa_s \omega_s n}{y_s^2} - \sum_{t=0}^S \frac{(2\omega_t - \delta_{st} \omega_{ad_s}) \omega_s n}{y_s y_t} \right) \\
&\stackrel{(37)}{=} \sum_{s=0}^S \frac{\omega_s n}{y_s} \left(\frac{\kappa_s + \omega_{ad_s}}{y_s} - \sum_{t=0}^S \frac{2\omega_t}{y_t} \right) \\
&= \left(\sum_{s=0}^S \frac{\omega_s n}{y_s} \right) \left(1 - \sum_{t=0}^S \frac{2\omega_t}{y_t} \right) \\
&\stackrel{(41)}{=} c_u \cdot \frac{2c_K}{n} \quad \square
\end{aligned}$$

The following lemma was shown in [1]:

Lemma 3 *Suppose that $\mathfrak{g} \equiv \mathfrak{u} \oplus \mathfrak{p}$ forms a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra, as in (29) with $\mathfrak{g}_0 = \mathfrak{u}$ and $\mathfrak{g}_1 = \mathfrak{p}$, which is related to the given orthogonal \mathfrak{u} -module-structure on $\mathfrak{u} \oplus \mathfrak{p}$ as follows:*

- *The Killing form of \mathfrak{g} coincides on $\mathfrak{p} \times \mathfrak{p}$ with the given orthogonal inner product, i.e. $\text{Tr}(ad_{\mathfrak{g}}(p_\alpha) ad_{\mathfrak{g}}(p_\beta)) = \delta_{\alpha, \beta} \quad \forall 1 \leq \alpha, \beta \leq n$.*
- *The restriction of the adjoint representation of \mathfrak{g} to \mathfrak{u} coincides with the given \mathfrak{u} -module structure of $\mathfrak{u} \oplus \mathfrak{p}$, i.e.*

$$\begin{aligned}
[u^i, u^j + p_\alpha]_{\mathfrak{g}} &= [u^i, u^j]_{\mathfrak{u}} + u^i(p_\alpha) \\
&= [u^i, u^j]_{\mathfrak{u}} + \sum_{\gamma} M_{\gamma\alpha}^i p_\gamma, \quad 1 \leq i, j \leq \dim \mathfrak{u}, \quad 1 \leq \alpha \leq n
\end{aligned} \tag{45}$$

where $[\cdot, \cdot]_{\mathfrak{g}}$ and $[\cdot, \cdot]_{\mathfrak{u}}$ denote the Lie products on \mathfrak{g} and \mathfrak{u} , respectively.

Then the Lie product on \mathfrak{p} is given by

$$[p_\alpha, p_\beta]_{\mathfrak{g}} = \sum_{i=1}^{\dim \mathfrak{u}} \frac{1}{y_{s(i)}} M_{\alpha\beta}^i u^i. \tag{46}$$

Proof: The invariance (3) of the Killing form means that

$$Tr(ad(u^i) ad([p_\alpha, p_\beta])) = Tr(ad([u^i, p_\alpha]) ad(p_\beta)). \quad (47)$$

Since $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_0$ by assumption, we must have $[p_\alpha, p_\beta] = \sum_{j=1}^{\dim \mathfrak{u}} X_{\alpha\beta}^j u^j$, where the $X_{\alpha\beta}^j$ are some constants, such that $X_{\alpha\beta}^j = -X_{\beta\alpha}^j$. These constants can be determined by equating the l.h.s. of (47)

$$\sum_{j=1}^{\dim \mathfrak{u}} X_{\alpha\beta}^j Tr(ad(u^i) ad(u^j)) = \sum_{j=1}^{\dim \mathfrak{u}} -X_{\alpha\beta}^j y_{s(i)} \delta_{ij} = -X_{\alpha\beta}^i y_{s(i)},$$

with the r.h.s. of (47)

$$Tr(ad([u^i, p_\alpha]) ad(p_\beta)) = \sum_{\gamma=1}^n M_{\gamma\alpha}^i Tr(ad(p_\gamma) ad(p_\beta)) = \sum_{\gamma=1}^n M_{\gamma\alpha}^i \delta_{\gamma\beta} = M_{\beta\alpha}^i.$$

It follows that

$$X_{\alpha\beta}^i = -\frac{1}{y_{s(i)}} M_{\beta\alpha}^i = \frac{1}{y_{s(i)}} M_{\alpha\beta}^i. \quad \square$$

Symmetric Space Theorem (Goddard, Nahm, Olive) *Let U be a compact Lie group with a faithful n -dimensional orthogonal representation on \mathfrak{p} . Consider the Lie algebra \mathfrak{u} of U as a subalgebra of $so(n)$ with the inclusion of \mathfrak{u} in $so(n)$ induced by the representation on \mathfrak{p} . Let c_K denote the central element of the coset Virasoro algebra on a level one representation space of $\tilde{\mathcal{L}}(so(n))$.*

Then c_K vanishes if and only if $\mathfrak{g} \equiv \mathfrak{u} \oplus \mathfrak{p}$ carries the structure of a $\mathbb{Z}/2\mathbb{Z}$ -graded Lie algebra with $\mathfrak{g}_0 = \mathfrak{u}$ and $\mathfrak{g}_1 = \mathfrak{p}$, which is related to the given orthogonal \mathfrak{u} -module-structure on $\mathfrak{u} \oplus \mathfrak{p}$ as in lemma 3.

Proof: The conditions of the theorem, in view of lemma 3, leave no freedom in defining the Lie product $[\cdot, \cdot]_{\mathfrak{g}}$, if it exists: This is because these conditions already uniquely determine a bilinear antisymmetric product on $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$, as follows:

$$[u^i, u^j]_{\mathfrak{g}} = [u^i, u^j]_{\mathfrak{u}}$$

$$\begin{aligned}
[u^i, p_\alpha]_{\mathfrak{g}} &= \sum_{\gamma=1}^n M_{\gamma\alpha}^i p_\gamma = -[p_\alpha, u^i]_{\mathfrak{g}} \\
[p_\alpha, p_\beta]_{\mathfrak{g}} &= \sum_{i=1}^{\dim \mathfrak{u}} \frac{1}{y_{s(i)}} M_{\alpha\beta}^i u^i
\end{aligned}$$

(cf. lemma 3). Therefore, the proof of the theorem boils down to showing, that c_K vanishes if and only if this bilinear antisymmetric product on \mathfrak{g} obeys the Jacobi identity.

It can easily be shown from the assumptions, that the Jacobi identity holds for any three elements, if at least one of them lies in \mathfrak{u} . Therefore, \mathfrak{g} is a Lie algebra iff the Jacobi identity holds on its \mathfrak{p} part, i.e. iff

$$0 = [p_\alpha, [p_\beta, p_\gamma]] + [p_\gamma, [p_\alpha, p_\beta]] + [p_\beta, [p_\gamma, p_\alpha]] = \sum_{\delta=1}^n J_{\alpha\beta\gamma\delta} p_\delta \quad \forall \alpha, \beta, \gamma, \quad (48)$$

where we used the equation

$$[p_\alpha, [p_\beta, p_\gamma]] = \sum_{\delta=1}^n \left(\sum_{i=1}^{\dim \mathfrak{u}} \frac{M_{\gamma\beta}^i M_{\delta\alpha}^i}{y_{s(i)}} \right) p_\delta ,$$

which follows from lemma 3.

Thus, the Jacobi identity (48) is equivalent to the condition

$$J_{\alpha\beta\gamma\delta} = 0 \quad \forall \alpha, \beta, \gamma, \delta. \quad (49)$$

But since the M^i are real, the $J_{\alpha\beta\gamma\delta}$ are also real. Therefore, each individual $J_{\alpha\beta\gamma\delta}$ vanishes if and only if the sum of the squares $(J_{\alpha\beta\gamma\delta})^2$ on the l.h.s. of (40) vanishes. But since by lemma 2 this sum is equal to $\frac{6}{n} c_{\mathfrak{u}} c_K$ and $c_{\mathfrak{u}}$ is always positive, it follows that (49) is equivalent to the vanishing of c_K :

$$c_K = 0, \quad \text{iff} \quad J_{\alpha\beta\gamma\delta} = 0 \quad \forall \alpha, \beta, \gamma, \delta. \quad (50)$$

□

The equivalence (50) was shown in [5] using the quark model construction. This result and the observation that (49) is equivalent to a Jacobi identity led to the symmetric space theorem [1].

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